The SABR/LIBOR Market Model
Pricing, Calibration and Hedging for Complex Interest-Rate Derivatives

Riccardo Rebonato
Kenneth McKay
and
Richard White
The SABR/LIBOR Market Model
The SABR/LIBOR Market Model
Pricing, Calibration and Hedging for Complex Interest-Rate Derivatives

Riccardo Rebonato
Kenneth McKay
and
Richard White

WILEY
A John Wiley and Sons, Ltd., Publication
Contents

Acknowledgements xi

1 Introduction 1

I The Theoretical Set-Up 7

2 The LIBOR Market Model 9

2.1 Definitions 10
2.2 The Volatility Functions 11
2.3 Separating the Correlation from the Volatility Term 12
2.4 The Caplet-Pricing Condition Again 14
2.5 The Forward-Rate/Forward-Rate Correlation 16
2.5.1 The Simple Exponential Correlation 16
2.5.2 The Multiplicative Correlation 17
2.6 Possible Shapes of the Doust Correlation Function 19
2.7 The Covariance Integral Again 21

3 The SABR Model 25

3.1 The SABR Model (and Why it is a Good Model) 25
3.2 Description of the Model 26
3.3 The Option Prices Given by the SABR Model 27
3.4 Special Cases 28
3.4.1 ATM Options 28
3.4.2 The Normal Case (\( \beta = 0 \)) 28
3.4.3 The Log-Normal Case (\( \beta = 1 \)) 29
3.5 Qualitative Behaviour of the SABR Model 29
3.5.1 Dependence on \( \sigma_0^T \) 29
3.5.2 Dependence on \( \beta \) 31
3.5.3 Dependence on \( \rho \) 33
3.5.4 Dependence on \( v \) 33
3.6 The Link Between the Exponent, \( \beta \), and the Volatility of Volatility, \( v \) 35
CONTENTS

3.7 Volatility Clustering in the (LMM)-SABR Model 37
3.8 The Market 40
  3.8.1 Analysis of $\sigma_0^T (\beta = 0.5)$ 40
  3.8.2 Analysis of $\nu^T (\beta = 0.5)$ 41
  3.8.3 Analysis of $\rho^T (\beta = 0.5)$ 43
3.9 How Do We Know that the Market has Chosen $\beta = 0.5$? 43
3.10 The Problems with the SABR Model 46
  3.10.1 Log-Normality of the Volatility Process 46
  3.10.2 Problems with the (Stochastic) CEV Process 47

4 The LMM-SABR Model 51
  4.1 The Equations of Motion 52
  4.2 The Nature of the Stochasticity Introduced by Our Model 53
  4.3 A Simple Correlation Structure 54
  4.4 A More General Correlation Structure 55
  4.5 Observations on the Correlation Structure 57
  4.6 The Volatility Structure 58
  4.7 What We Mean by Time Homogeneity 59
  4.8 The Volatility Structure in Periods of Market Stress 59
  4.9 A More General Stochastic Volatility Dynamics 63
  4.10 Calculating the No-Arbitrage Drifts 64
    4.10.1 Preliminaries 64
    4.10.2 Standard LIBOR and LIBOR in Arrears 70
    4.10.3 LIBOR in Arrears: The Volatility Drift 73
    4.10.4 The Drifts in the General Case of Several Forward Rates 74
    4.10.5 Volatility Drifts in the Swap Measure 75

II Implementation and Calibration 79

5 Calibrating the LMM-SABR Model to Market Caplet Prices 81
  5.1 The Caplet-Calibration Problem 81
  5.2 Choosing the Parameters of the Function, $g(\cdot)$, and the Initial Values, $k_0^T$ 83
  5.3 Choosing the Parameters of the Function $h(\cdot)$ 84
  5.4 Choosing the Exponent, $\beta$, and the Correlation, $\phi_{SABR}$ 88
  5.5 Results 88
  5.6 Calibration in Practice: Implications for the SABR Model 91
    5.6.1 Looking at Caplets in Isolation 91
    5.6.2 Looking at Caplets and Swaptions Together 95
  5.7 Implications for Model Choice 99

6 Calibrating the LMM-SABR Model to Market Swaption Prices 101
  6.1 The Swaption Calibration Problem 101
CONTENTS

6.2 Swap Rate and Forward Rate Dynamics 102
6.3 Approximating the Instantaneous Swap Rate Volatility, $S_t$ 104
6.4 Approximating the Initial Value of the Swap Rate Volatility, $\Sigma_0$ (First Route) 105
6.5 Approximating $\Sigma_0$ (Second Route) and the Volatility of Volatility of the Swap Rate, $\nu$ 106
6.6 Approximating the Swap-Rate/Swap-Rate-Volatility Correlation, $R_{SABR}$ 108
6.7 Approximating the Swap Rate Exponent, $B$ 108
6.8 Results 109
   6.8.1 Comparison between Approximated and Simulation Prices 109
   6.8.2 Comparison between Parameters from the Approximations and the Simulations 117
6.9 Conclusions and Suggestions for Future Work 118
6.10 Appendix: Derivation of Approximate Swap Rate Volatility 118
6.11 Appendix: Derivation of Swap-Rate/Swap-Rate-Volatility Correlation, $R_{SABR}$ 120
6.12 Appendix: Approximation of $dS_t/S_t$ 122

7 Calibrating the Correlation Structure 125
7.1 Statement of the Problem 125
7.2 Creating a Valid Model Matrix 126
   7.2.1 First Strategy, Stage 1: Diagonalize $P$ 128
   7.2.2 First Strategy, Stage 2: Analytic Optimization of $c_i$ 128
   7.2.3 Second Strategy: Optimizing over Angles 129
7.3 A Case Study: Calibration Using the Hypersphere Method 131
7.4 Which Method Should One Choose? 137
7.5 Appendix 138

III Empirical Evidence 141

8 The Empirical Problem 143
8.1 Statement of the Empirical Problem 143
8.2 What Do We Know from the Literature? 145
8.3 Data Description 148
8.4 Distributional Analysis and Its Limitations 150
8.5 What is the True Exponent $\beta$? 153
8.6 Appendix: Some Analytic Results 155

9 Estimating the Volatility of the Forward Rates 159
9.1 Expiry Dependence of Volatility of Forward Rates 160
9.2 Direct Estimation 162
9.3 Looking at the Normality of the Residuals 164
9.4 Maximum-Likelihood and Variations on the Theme 171
9.5 Information About the Volatility from the Options Market 175
9.6 Overall Conclusions 178

10 Estimating the Correlation Structure 181
10.1 What We are Trying to Do 181
10.2 Some Results from Random Matrix Theory 182
10.3 Empirical Estimation 185
10.4 Descriptive Statistics 185
  10.4.1 The Forward-Rate/Forward-Rate Correlation Matrix 185
  10.4.2 The Forward-Rate/Volatility Correlation Block 187
  10.4.3 The Volatility/Volatility Correlation Matrix 188
10.5 Signal and Noise in the Empirical Correlation Blocks 188
  10.5.1 The Forward-Rate/Forward-Rate Correlation Matrix 188
  10.5.2 The Volatility/Volatility Correlation Matrix 190
  10.5.3 The Forward-Rate/Volatility Correlation Block 190
10.6 What Does Random Matrix Theory Really Tell Us? 190
10.7 Calibrating the Correlation Matrices 191
  10.7.1 The Fitting Procedure 192
  10.7.2 Results 192
10.8 How Much Information Do the Proposed Models Retain? 195
  10.8.1 Eigenvalues of the Correlation Blocks 195
  10.8.2 Eigenvalues of Differences in the Correlation Blocks 196
  10.8.3 Entropy Measures 198
  10.8.4 The Forward-Rate/Volatility Correlation Block 202

IV Hedging 203

11 Various Types of Hedging 205
11.1 Statement of the Problem 205
11.2 Three Types of Hedging 206
  11.2.1 In- and Out-of-Model Hedging 206
  11.2.2 Functional-Dependence Hedging 207
11.3 Definitions 210
11.4 First-Order Derivatives with Respect to the Underlyings 211
  11.4.1 Delta Hedging 211
  11.4.2 Vega Hedging 213
11.5 Second-Order Derivatives with Respect to the Underlyings 214
  11.5.1 Vanna and Volga 214
11.6 Generalizing Functional-Dependence Hedging 215
11.7 How Does the Model Know about Vanna and Volga? 219
11.8 Choice of Hedging Instrument 220
12 Hedging against Moves in the Forward Rate and in the Volatility 221
  12.1 Delta Hedging in the SABR-(LMM) Model 221
  12.2 Vega Hedging in the SABR-(LMM) Model 229
13 (LMM)-SABR Hedging in Practice: Evidence from Market Data 231
  13.1 Purpose of this Chapter 231
  13.2 Notation 231
    13.2.1 Estimation of the Unobservable Volatility 232
    13.2.2 Tests of the Hedging Performance of the SABR Model 233
    13.2.3 Tests of the Hedging Performance of the LMM-SABR Model 233
  13.3 Hedging Results for the SABR Model 234
  13.4 Hedging Results for the LMM-SABR Model 243
  13.5 Conclusions 245
14 Hedging the Correlation Structure 247
  14.1 The Intuition Behind the Problem 247
  14.2 Hedging the Forward-Rate Block 249
  14.3 Hedging the Volatility-Rate Block 251
  14.4 Hedging the Forward-Rate/Volatility Block 253
  14.5 Final Considerations 254
15 Hedging in Conditions of Market Stress 257
  15.1 Statement of the Problem 257
  15.2 The Volatility Function 259
  15.3 The Case Study 260
  15.4 Hedging 261
    15.4.1 The Normal-to-Normal State Transition 261
    15.4.2 The Normal-to-Excited Transition 263
    15.4.3 Normal-to-Unknown Transition 265
    15.4.4 Starting from the Excited State 266
  15.5 Results 266
    15.5.1 Hedging Results for the Normal-to-Normal Transition 267
    15.5.2 Hedging Results for the Normal-to-Excited Transition 267
  15.6 Are We Getting Something for Nothing? 270

References 271

Index 275
Acknowledgements

It is a pleasure to acknowledge the help provided by many colleagues and friends. In particular, the advice and suggestions of Paul Doust, Andrei Pogudin, Jian Chen, Raphael Albrecht, Dhermider Kainth and Michael Dogwood have been of great help. This book is much the better thanks to them.

We are grateful to John Wiley for agreeing to publish this book, and for the enthusiasm they have shown for the project. Caitlin Cornish has been a most efficient and supportive commissioning editor.

Finally, two of us (RR and KM) cannot help feeling some pangs of envy towards our third co-author, Richard. Unfortunately for us, but probably wisely for him, a few months into the project Richard decided to take a year off to tour the world with his girlfriend. We suspect that the pleasures of proofreading and reference checking may have played a part in making trekking through Siberia appear more attractive than it is normally cracked up to be. Be that as it may, his contribution to this book has been so important that, proofreading or no proofreading, he has earned full authorship, and we feel proud to have him as third co-author. (Just don’t do this again, Richard.)
Chapter 1

Introduction

All models are wrong, but some models are useful

We present in this book a financially motivated extension of the LIBOR market model that reproduces for all strikes and maturities the prices of the plain-vanilla hedging instruments (swaptions and caplets) produced by the SABR model. In other words, our extension of the LIBOR market model accurately recovers in a financially motivated manner the whole of the SABR smile surface.

As the SABR model has become the ‘market standard’ for European options, just the recovery of the smile surface by a dynamic model could be regarded as a useful achievement in itself. However, we have tried to do more. As we have stressed in the opening sentences, we have tried to accomplish this task in a way that we consider financially justifiable.

Our reason for insisting on financial reasonableness is not (just) an aesthetic one. We believe that the quality of a derivatives model should be judged not just on the basis of its ability to price today’s hedging instruments, but also on the basis of the quality of the hedges it suggests. We believe that these hedges can be good only if the model is rooted in empirical financial reality. The ‘empirical financial reality’ of relevance for the pricing and hedging of complex derivatives is the dynamics of the smile surface. We explain below why we believe that this is the case.

We are therefore not just offering yet another model. We present a ‘philosophy’ of option pricing that takes into account the realities of the industry needs (e.g., the need to calibrate as accurately as possible to the plain-vanilla reference hedging instruments, the need to obtain prices and hedges in reasonable time) while reproducing a realistic future evolution of the smile surface (our ‘financial reality’).

Until recently choosing between fitting today’s prices very accurately and being respectful of ‘financial reality’ (given our meaning of the term) entailed making hard choices. For instance, some approaches, such as local-volatility modelling (see, e.g., Dupire (1994), Derman and Kani (1994)), fulfilled (by construction) very well the first set of requirements (perfect fitting of today’s smile). This made local volatility models very popular with some traders. Yet, the dynamics of the smile these models implied were completely wrong. Indeed, the SABR model, which constitutes the starting point for our extension, was introduced to remedy the wrong dynamics imposed by the local-volatility framework.