Researchers of continuum mechanics of solids and structures and structural analysts in industry will find this book extremely insightful. It will also be of great interest to graduate and postgraduate students of mechanical, civil and aerospace engineering.

Key features:
- compares classical and modern approaches to beam theory, including classical well-known results related to Euler-Bernoulli and Timoshenko beam theories
- pays particular attention to typical applications related to bridge structures, aircraft wings, helicopters and propeller blades
- provides a number of numerical examples including typical Aerospace and Civil Engineering problems
- proposes many benchmark assessments to help the reader implement the CUF if they wish to do so
- accompanied by a companion website hosting dedicated software MUL2 that is used to obtain the numerical solutions in the book, allowing the reader to reproduce the examples given in the book as well as to solve other problems of their own

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After earning two degrees (Aeronautics, 1986, and Aerospace Engineering, 1988) at the Politecnico di Torino, Erasmo Carrera received his PhD degree in Aerospace Engineering jointly at the Politecnico di Milano, Politecnico di Torino, and Università di Pisa in 1991. He began working as a Researcher in the Department of Aerospace for the Politecnico di Torino in 1992 where he held courses on Missiles and Aerospace Structure Design, Plates and Shells, and the Finite Element Method. He became Associate Professor of Aerospace Structures and Computational Aeroelasticity in 2000, and Full Professor at the Politecnico di Torino in 2011. He has visited the Institute für Statik und Dynamik, Universität Stuttgart twice, the first time as a PhD student (six months in 1991) and then as Visiting Scientist under a GKKS Grant (18 months in 1995–1996). In the summers of 1996, 2003 and 2009, he was Visiting Professor at the ESM Department of Virginia Tech, at SUPMECA in Paris (France) and at the CRP TUDOR in Luxembourg, respectively. His main research topics are: inflatable structures, composite materials, finite elements, plates and shells, postbuckling and stability, smart structures, thermal stress, aeroelasticity, multibody dynamics, and the design and analysis of non-classical lifting systems. He is author of more than 300 articles on these topics, many of which have been published in international journals. He serves as referee for international journals, and as a contributing editor for Mechanics of Advanced Materials and Structures, Composite Structures and Journal of Thermal Stress.

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Preface

Beam models have made it possible to solve a large number of engineering problems over the last two centuries. Early developments, based on kinematic intuitions (bending theories), by pioneers such as Leonardo da Vinci, Euler, Bernoulli, Navier, and Barre de Saint Venant, have permitted us to consider the most general three-dimensional (3D) problem as a one-dimensional (1D) problem in which the unknowns only depend on the beam-axis position. These early theories are known as engineering beam theories (EBTs) or the Euler–Bernoulli beam theory (EBBT). Recent historical reviews have proposed that these theories should be referred to as the DaVinci–Euler–Bernoulli beam theory (DEBBT). The drawbacks of EBT are due to the intrinsic decoupling of bending and torsion (cross-section warping is not addressed by EBT) as well as to the difficulties involved in evaluating the additional five (normal and shear) stress components that are not provided by the Navier formula. Many torsion-beam theories which are effective for different types of beam sections are known. Many refinements of original EBT kinematics have been proposed. Amongst these, the one attributed to Timoshenko in which transverse shear deformations are included should be mentioned. The other refined theories mentioned herein are those by Vlasov and by Wagner, both of which lead to improved strain/stress field descriptions.

Over the last few decades, computational methods, in particular the finite element method, have made the use of classical beam theories much more successful and attractive. The possibility of solving complex framed structures with very different boundary conditions (mechanical and geometrical) has made it possible to analyze many complex problems involving thousands of degrees of freedom (DOFs) with acceptable accuracy. However, the difficulty of obtaining a complete stress/strain field in those sections with complex geometries or thin walls still remains an open question which can be addressed by refined and advanced beam theories.

During the last decade, the first author of this book proposed the Carrera Unified Formulation (CUF), which was first applied to plates and shells and then
recently extended to beams. The CUF permits one to develop a large number of beam theories with a variable number of displacement unknowns by means of a concise notation and by referring to a few fundamental nuclei. Higher-order beam theories can easily be implemented on the basis of the CUF, and the accuracy of a large variety of beam theories can be established in a hierarchical and/or axiomatic vs. asymptotic sense. A modern form of beam theories can therefore be constructed in a hierarchical manner. The number of unknown variables is a free parameter of the problem. A 3D stress/strain field can be obtained by an appropriate choice of these variables for any type of beam problem: compact sections, thin-walled sections, bending, torsion, shear, localized loadings, static and dynamic problems.

This book details classical and modern beam theories. Accuracy of the known theories is established by using the modern technique in the CUF. Various beam problems, in particular beam sections from civil to aerospace applications (wing airfoils), are considered in static and dynamic problems. Numerical results are obtained using the MUL2 software, which is available on the web site www.mul2.com.

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Introduction

A brief introduction to the contents of the book is given here together with an overview of the milestone contributions to beam structure analysis.

Why another book on beams?

There is no need for another book on beam theories. Many books are, in fact, available, which have been written by some of the most eminent and talented scientists in the theory of elasticity and structures. It would be extremely difficult to write a better book. So, why a new book on beam theories? The reason is the following: this book presents a method to deal with beam theories that has never been considered before. As will be explained in the following chapters, the method introduced by the first author over the last decade for plates and shells is applied here to beams to build a large class of 1D (beam) hierarchical (variable kinematic) theories, which are based on automatic techniques to build governing equations and/or finite element matrices. The resulting theories permit one to deal with any section geometries subjected to any loading conditions and, at the same time, to reach quasi-3D solutions. Such results make the present book unique.

Review of historical contributions

Beam theories are extensively used to analyze the structural behavior of slender bodies, such as columns, arches, blades, aircraft wings, and bridges. The main advantage of beam models is that they reduce the 3D problem to a set of variables that only depends on the beam-axis coordinate. The 1D structural elements obtained are simpler and computationally more efficient than 2D (plate/shell) and 3D (solid) elements. This feature makes beam theories very attractive for the static and dynamic analysis of structures.

The classical, most frequently employed theories are those by Euler–Bernoulli (Bernoulli, 1751; Euler, 1744), de Saint-Venant (1856a,b), and Timoshenko (1921,