

# Statistical Size Distributions in Economics and Actuarial Sciences

CHRISTIAN KLEIBER

Universität Dortmund, Germany

SAMUEL KOTZ

The George Washington University



A JOHN WILEY AND SONS, INC., PUBLICATION



Statistical Size Distributions in  
Economics and Actuarial Sciences

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Published by John Wiley & Sons, Inc., Hoboken, New Jersey.  
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***Library of Congress Cataloging-in-Publication Data:***

Kleiber, Christian, 1966-

Statistical size distributions in economics and actuarial sciences/Christian Kleiber, Samuel Kotz.  
p. cm.—(Wiley series in probability and statistics)

Includes bibliographical references and index.

ISBN 0-471-15064-9 (cloth)

1. Distribution (Economic theory)
3. Economics, Mathematical.
4. Insurance—Mathematics. I. Kotz, Samuel. II. Title. III. Series.

HB523.K55 2003

339.2'2—dc21

2003041140

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

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# Preface

This is a book about money, but it will not help you very much in learning how to make money. Rather, it will instruct you about the distribution of various kinds of income and their related economic size distributions. Specifically, we have painstakingly traced the numerous statistical models of income distribution, from the late nineteenth century when Vilfredo Pareto developed a bold and astonishing model for the distribution of personal income until the latest models developed some 100 years later. Our goal was to review, compare, and somehow connect all these models and to pinpoint the unfortunate lack of coordination among various researchers, which has resulted in the duplication of effort and waste of talent and to some extent has reduced the value of their contributions. We also discuss the size distributions of loss in actuarial applications that involve a number of distributions used for income purposes. An impatient reader may wish to consult the list of distributions covered in this book and their basic properties presented in Appendix C.

The task of compiling this interdisciplinary book took longer and was more arduous than originally anticipated. We have tried to describe the distributions outlined here within the context of the personalities of their originators since in our opinion the personality, temperament, and background of the authors cited did affect to some extent the nature and scope of their discoveries and contributions.

We hope that our readers come to regard this book as a reliable source of information and we gladly welcome all efforts to bring any remaining errors to our attention.

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# Acknowledgments

The authors are indebted to various researchers around the globe—too numerous to be mentioned individually—for generously providing us with preprints, reprints, and useful advice.

Special thanks are due to Professor Giovanni Maria Giorgi for writing four biographies of leading contributors to the field, to Professors Camilo Dagum and Gabriele Stoppa for reading parts of the original manuscript and offering us the most valuable suggestions and comments, to Professor Constance van Eeden and Meike Gebel for translations from the Dutch and Italian, respectively, and to Professor Fiorenzo Mornati for supplying important not easily accessible information about Vilfredo Pareto. The first author would also like to thank Professor Walter Krämer for his support over (by now) many years.

All of the graphs in this book were generated using the R statistical software package (<http://www.r-project.org/>), the GNU implementation of the S language.



## CHAPTER ONE

# Introduction

*Certum est quia impossibile est.* TERTULLIAN, 155/160 A.D.—after 220 A.D.

*This book is devoted to the parametric statistical distributions of economic size phenomena of various types—a subject that has been explored in both statistical and economic literature for over 100 years since the publication of V. Pareto’s famous breakthrough volume Cours d’économie politique in 1897. To the best of our knowledge, this is the first collection that systematically investigates various parametric models—a more respectful term for distributions—dealing with income, wealth, and related notions. Our aim is marshaling and knitting together the immense body of information scattered in diverse sources in at least eight languages. We present empirical studies from all continents, spanning a period of more than 100 years.*

*We realize that a useful book on this subject matter should be interesting, a task that appears to be, in T. S. Eliot’s words, “not one of the least difficult.” We have tried to avoid reducing our exposition to a box of disconnected facts or to an information storage or retrieval system. We also tried to avoid easy armchair research that involves computerized records and heavy reliance on the Web.*

*Unfortunately, the introduction by its very nature is always somewhat fragmentary since it surveys, in our case rather extensively, the content of the volume. After reading this introduction, the reader could decide whether continuing further study of the book is worthwhile for his or her purposes. It is our hope that the decision will be positive. To provide a better panorama, we have included in the Appendix brief biographies of the leading players.*

### 1.1 OUR AIMS

The modeling of economic size distributions originated over 100 years ago with the work of Vilfredo Pareto on the distribution of income. He apparently was the first to