MATHEMATICS OF SHAPE DESCRIPTION
A Morphological Approach to Image Processing and Computer Graphics
To
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and Kazuko
To the memory of Pijush K. Ghosh

The doors to knowledge were opened to me at an early age by my father. He taught me that the thirst for knowledge is unquenchable. He was like my very own magician, who made learning creative and fun. Today he is no longer by my side to guide me, but the fulfillment of his dream in the form of this book brings me immense joy. He is alive to me in the pages of this book, and this book is a ray of light in the darkness he has left behind in my life.

Daddy’s Little Girl ...
Nairita Ghosh
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Foreword

The computer description of shape and the computer manipulation of shape is complex simply because shape itself is complex. Of course, if the world of shape were limited to the Euclidean shapes, there would be no such complexity. However, shape includes all the varieties of biological shapes: from the shapes of trees and their leaves to fish, animals, flowers, and plants – and also natural shapes, such as those of coastlines, and of rocks and crystals.

Mathematical morphology is the mathematical study of shapes through a particular algebra of operations, known as the Minkowski set operations. Here, a shape can be thought of in the most general way possible, as a set of points in two or three dimensions. To fully understand the nature of the algebra of mathematical morphology requires: (1) an understanding of what an axiom system actually provides; (2) fluency in a variety of concepts associated with sets, including the set builder notation in mathematics; and (3) fluency in the concepts of algebraic structures. It is in this setting, formulated by Professor Deguchi, that the particulars of the concepts of mathematical morphology can most fully be appreciated.

Mathematics of Shape Description is the first book to devote half of its pages, in a tutorial fashion, to the basic background and/or essential preliminary concepts that lead up to the definitions of the mathematical morphological operators. This treatment of mathematical morphology simultaneously handles the discrete and the continuous domains, and is based on the mathematical morphology papers of Pijush Ghosh.

I knew Pijush Ghosh in the early 1990s, when he came to visit my laboratory at the University of Washington. His knowledge and understanding of mathematical morphology operations and what could be done with them, and what structures to use to implement continuous domain morphology in a computer program, was thorough and complete. I learned a great deal from him. He was a beautiful person, with a wonderful mind. He passed from this world prematurely, at an early age, only a few years after he returned to India, and he is greatly missed.

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