AN INTRODUCTION TO
COMPUTATIONAL FLUID
MECHANICS BY EXAMPLE
AN INTRODUCTION TO COMPUTATIONAL FLUID MECHANICS BY EXAMPLE

Sedat Biringen and Chuen-Yen Chow
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To our spouses,

to our children,

and to the memory of our parents, who gave us the spirit for intellectual
and creative pursuit
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This book is based on the original textbook by C.-Y. Chow entitled *An Introduction to Computational Fluid Mechanics*, adopted and used by both authors in Computational Fluid Dynamics/Mechanics (CFDM) courses they have taught at the University of Colorado at Boulder and at the University of New Hampshire at Durham (SB). The original text was written in a highly accessible manner with senior undergraduate and first-year graduate students in mind and occasionally has been benefited by researchers in mechanical and aerospace engineering disciplines. Over the 25 years since the original publication, the field of CFDM has seen many changes, evolutions, and advances in algorithmic developments as well as in computer software/hardware. The new book incorporates some of the modern algorithmic developments into the solution techniques implemented in the vast number of examples provided in the text. Concurrently, we tried to widen the scope of the applications to include examples relevant to other engineering disciplines to make the text attractive and useful for a larger audience.

We revised the computer programs included in the original text and converted all the programs to MATLAB, one of the most widely adopted computer languages in engineering education. The new MATLAB programs are available online on the book’s web site (www.wiley.com/go/Biringen). The reader is expected to have a working knowledge of MATLAB programming basics. The core-scope of the new book was expanded to include more up-to-date solution methods for the Navier-Stokes equations, including fractional step time-advancement and pseudo-spectral methods. In summary, we expect the new text to create a unique niche because of its hands-on approach and practical content and to have wide appeal in the classroom as well as in the research environment.

The pedagogical approach used in this book follows the path of the original and focuses on teaching by the study of actual examples from fluid mechanics. It is our belief that building up from worked examples and providing a hands-on approach allows students to implement simple codes as a very effective means of teaching complex material. This approach is the unique aspect of our book, primarily as a teaching instrument. In addition, more advanced solution procedures can be constructed based on the provided solvers.

The contents of the current book follow closely the contents of Chow’s book, with additions relevant to the solution of the full Navier-Stokes equations.
Almost all solution methods presented are based on finite differences. The book should be suitable for a two-semester course in computational fluid mechanics, or topics can be selected for a one-semester course at the beginning graduate level. There is sufficient material for a more advanced course, or selected topics can be included as a supplement to traditional textbooks for courses in fluid mechanics at undergraduate or graduate levels.

We emphasize that this is predominantly an introductory book that teaches how to implement computational methods in fluid mechanics applications and not a book on numerical computation/analysis. It deals with flow problems that either have to be solved numerically or can be made much simpler with the help of computational tools. Numerical methods and algorithms are presented simply as tools implemented to solve physical problems; detailed analyses and critical evaluation of these techniques are not attempted. Of course, several methods exist for numerically integrating a given ordinary or partial differential equation; the numerical methods adopted in this book are only the simpler ones or the commonly used ones with which the authors have intimate experience and are by no means the complete spectrum of available methods. The book also does not cover in detail more advanced topics such as mesh generation and solution methods for the full compressible Navier-Stokes equation; also omitted are more advanced techniques such as multigrid methods, and other elliptic solvers.

The authors have benefited from interactions with many bright and capable students who have contributed to this book in various ways to improve the content. Particularly we (SB) thank Dr. Gökhan Danabaşoğlu for his many contributions when he was a teaching assistant for Advanced Computational Fluid Mechanics at the University of Colorado many years ago, Dr. Manuel Barcelos for his assistance in translating the FORTRAN codes from Chow’s book to MATLAB codes available on the web site for the book (www.wiley.com/go/biringen), and Scott Waggy for his programming assistance.

The first author (SB) would like to emphasize that the authorship order was designated alphabetically by the graceful insistence of C.-Y. Chow, who came out of his comfortable retirement to play a very active role to complete this project. All the successes of this book belong to him, and the first author will willingly shoulder the responsibility for all the potential shortcomings.
FLOW TOPICS GOVERNED BY ORDINARY DIFFERENTIAL EQUATIONS: INITIAL-VALUE PROBLEMS

The numerical solution of initial-value problems that involve nonlinear ordinary differential equations is considered in this chapter. In Section 1.1 some numerical methods, especially the Runge-Kutta methods, are introduced for solving the first- and second-order equations. They are applied in Section 1.2 for finding the motion of a free-falling sphere through air and in Section 1.3 to simulate the motions of a simple pendulum and an aeroelastic system.

To extend the applications from one-dimensional to two-dimensional motions, Runge-Kutta formulas for solving simultaneous second-order equations are deduced in Section 1.4. Simultaneously, we have implemented MATLAB initial value solver ODE45 in the programs developed in this chapter and elsewhere in the book. After the motion of a spherical projectile in the presence of a fluid has been computed, the numerical integration procedure of Section 1.5 is combined with the half-interval method to find the maximum range of such a body. Section 1.6 deals with the computation of the trajectory of a glider, and Section 1.7 is an example from aerodynamics concerning the vortex sheet trailing behind a finite wing.

1.1 NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS: INITIAL-VALUE PROBLEMS

Consider the simplest case of a first-order ordinary differential equation having the general form

\[
\frac{dx}{dt} = f(x, t) \tag{1.1.1}
\]

where \( f \) is an analytic function. If, at a starting point \( t = t_0 \), the function \( x \) has a given value \( x_0 \), it is desired to find \( x(t) \) for \( t > t_0 \) that satisfies both (1.1.1)
and the prescribed initial condition. Such a problem is called an initial-value problem.

To solve the problem numerically, the axis of the independent variable is usually divided into evenly spaced small intervals of width \( h \) whose end points are situated at

\[
t_i = t_0 + ih, \quad i = 0, 1, 2, \ldots
\]

(1.1.2)

The solution evaluated at the point \( t_i \) is denoted by \( x_i \). Thus, by using a numerical method, the continuous function \( x(t) \) is approximated by a set of discrete values \( x_i, \ i = 0, 1, 2, \ldots \), as sketched in Fig. 1.1.1. Since \( h \) is small and \( f \) is an analytic function, the solution at any point can be obtained by means of a Taylor’s series expansion about the previous point:

\[
x_{i+1} \equiv x(t_{i+1}) = x(t_i + h) \\
= x_i + h \left( \frac{dx}{dt} \right)_i + \frac{h^2}{2!} \left( \frac{d^2x}{dt^2} \right)_i + \frac{h^3}{3!} \left( \frac{d^3x}{dt^3} \right)_i + \cdots
\]

(1.1.3)

\[
= x_i + hf_i + \frac{h^2}{2!}f_i' + \frac{h^3}{3!}f_i'' + \cdots
\]

where \( f_i^n \) denotes \( \frac{d^n f}{dt^n} \) evaluated at \((x_i, t_i)\). \( f \) is generally a function of both \( x \) and \( t \), so that the first-order derivative is obtained according to the formula

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt} \frac{\partial f}{\partial x}
\]

Higher-order derivatives are obtained by using the same chain rule.

---

**FIGURE 1.1.1** Numerical solution of an ordinary initial-value problem.
Alternatively (1.1.3) can be rewritten as

\[ x_{i+1} = x_i + \Delta x_i \]  

(1.1.4)

where

\[ \Delta x_i = hf_i + \frac{h^2}{2!}f_i' + \frac{h^3}{3!}f_i'' + \cdots \]  

(1.1.5)

Starting from \( i = 0 \) with \( x_0 \) given and \( \Delta x_0 \) computed based on any desired number of terms in (1.1.5), the value of \( x_1 \) is first calculated. Then, by letting \( i = 1, 2, \) etc., in (1.1.4), the values of \( x_2, x_3, \) etc., are obtained successively. Theoretically, if the number of terms retained in (1.1.5) increases indefinitely, the numerical result from this marching scheme approaches the exact solution. In reality, however, it is not permissible to do so, and the series has to be cut off after a certain finite number of terms. For example, if two terms are retained on the right-hand side of (1.1.5) in computing \( \Delta x_i \), the value of \( x_{i+1} \) so obtained is smaller than the exact value by an amount \( (h^3/3!)f_i'' + (h^4/4!)f_i''' + \cdots \). For small \( h \) the first term is dominant. We may say that the error involved in this numerical calculation is of the order of \( h^3 f_i'' \), or simply \( O(h^3 f_i'') \). This is the truncation error that results from taking a finite number of terms in an infinite series.

It is termed Euler’s method when only one term is used on the right-hand side of (1.1.5). The truncation error is \( O(h^2 f_i') \), and the method should not be used if accuracy is demanded in the result.

It is impractical to use Taylor’s series expansion method if \( f \) is a function that has complicated derivatives. Furthermore, because of the dependence of the series on the derivatives of \( f \), a generalized computer program cannot be constructed for this method. The \( n \)th-order Runge-Kutta method is a commonly used alternative. Computations in this method require the evaluation of the function, \( f \), instead of its derivatives, with properly chosen arguments; the accuracy is equivalent to that with \( n \) terms retained in the series expansion (1.1.5). The second-order Runge-Kutta formulas are

\[ x_{i+1} = x_i + hf(x_i + \frac{1}{2} \Delta x_i, t_i + \frac{1}{2} h) \]  

(1.1.6)

where

\[ \Delta x_i = hf(x_i, t_i) \]  

(1.1.7)

For better results the following fourth-order Runge-Kutta formulas are usually employed:

\[ x_{i+1} = x_i + \frac{1}{6}(\Delta_1 x_i + 2\Delta_2 x_i + 2\Delta_3 x_i + \Delta_4 x_i) \]  

(1.1.8)

in which the increments are computed in the following order:

\[ \Delta_1 x_i = hf(x_i, t_i) \]

\[ \Delta_2 x_i = hf(x_i + \frac{1}{2} \Delta_1 x_i, t_i + \frac{1}{2} h) \]

\[ \Delta_3 x_i = hf(x_i + \frac{1}{2} \Delta_2 x_i, t_i + \frac{1}{2} h) \]

\[ \Delta_4 x_i = hf(x_i + \Delta_3 x_i, t_i + h) \]  

(1.1.9)